

Test of Goodness of Fit

Lecture 48

Sections 14.1 - 14.3

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Outline

- 1 Introduction
- 2 The Goodness-of-Fit Test
- 3 The Chi-Square Statistic
- 4 The Chi-Square Distribution
- 5 Goodness-of-Fit Test on the TI-83
- 6 One Last Comment
- 7 Assignment

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- But what do we do if we have more than two categories?
- For example,
 - Republican, Democrat, Independent.
 - Christian, Muslim, Jewish, Atheist.
 - Strongly approve, Approve, Disapprove, Strongly disapprove, No opinion.

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The Hypotheses

Example (Goodness-of-Fit Test)

- I would like to test whether a die is fair.
- I rolled the die 90 times and obtained the following results

| | | | | | | |
|-----------|---|----|----|----|----|----|
| Number | 1 | 2 | 3 | 4 | 5 | 6 |
| Frequency | 9 | 15 | 19 | 14 | 18 | 15 |

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- Does the die appear to be fair?

The Hypotheses

- To answer this question, we must return to the same question that we have asked many times already.
- Is the difference between what we observed and what we would expect to observe from a fair die small enough that we can reasonably attribute it to randomness?

The Hypotheses

- The null hypothesis specifies the probability (or proportion) for each category.
- The alternative hypothesis simply states that H_0 is false.

The Hypotheses

Example (Steps 1 and 2)

- (1) $H_0 : p_1 = 1/6, p_2 = 1/6, p_3 = 1/6, p_4 = 1/6, p_5 = 1/6, p_6 = 1/6.$
 $H_1 : H_0$ is false.
- (2) $\alpha = 0.05.$

Expected Counts

Definition (Observed and Expected Counts)

The **observed counts** are the counts that were actually observed in the sample. The **expected counts** are the counts that one would expect to observe if the null hypothesis were true.

Expected Counts

- To find the expected counts, we apply the hypothetical proportions to the sample size.
- For example, the hypothetical proportion for rolling a 1 is $1/6$, so we compute $1/6$ of 90:

$$\frac{1}{6} \times 90 = 15.$$

- Do *not* round the values off to whole numbers.

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The Test Statistic

- Make a chart showing both the observed counts and the expected counts (in parentheses).

| Number | 1 | 2 | 3 | 4 | 5 | 6 |
|------------|------|------|------|------|------|------|
| Observed | 9 | 15 | 19 | 14 | 18 | 15 |
| (Expected) | (15) | (15) | (15) | (15) | (15) | (15) |

- The expected counts are based on the hypothetical proportions.

The Test Statistic

- Denote the observed counts by O and the expected counts by E .
- Define the **chi-square (χ^2) statistic** to be

$$\chi^2 = \sum_{\text{all cells}} \frac{(O - E)^2}{E}.$$

The Hypotheses

Example (Step 3)

$$(3) \chi^2 = \sum_{\text{all cells}} \frac{(O - E)^2}{E}.$$

The Value of the Test Statistic

- Clearly, if *all* of the deviations $O - E$ are small, then χ^2 will be small.
- But if *even a few* of the deviations $O - E$ are large, then χ^2 will be large.

The Value of the Test Statistic

Example (Step 4)

(4)

| Number | 1 | 2 | 3 | 4 | 5 | 6 |
|------------|------|------|------|------|------|------|
| Observed | 9 | 15 | 19 | 14 | 18 | 15 |
| (Expected) | (15) | (15) | (15) | (15) | (15) | (15) |

$$\begin{aligned}\chi^2 &= \frac{(9 - 15)^2}{15} + \frac{(15 - 15)^2}{15} + \frac{(19 - 15)^2}{15} \\ &+ \frac{(14 - 15)^2}{15} + \frac{(18 - 15)^2}{15} + \frac{(15 - 15)^2}{15} \\ &= \frac{36}{15} + \frac{0}{15} + \frac{16}{15} + \frac{1}{15} + \frac{9}{15} + \frac{0}{15} \\ &= \frac{62}{15} \\ &= 4.133.\end{aligned}$$

Compute the p -Value

- The p -value for this example is the likelihood of observing a χ^2 value as large as 4.133, if the die is fair.
- To find that value, we need to know something about the distribution of χ^2 for a fair die.

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Chi-Square Degrees of Freedom

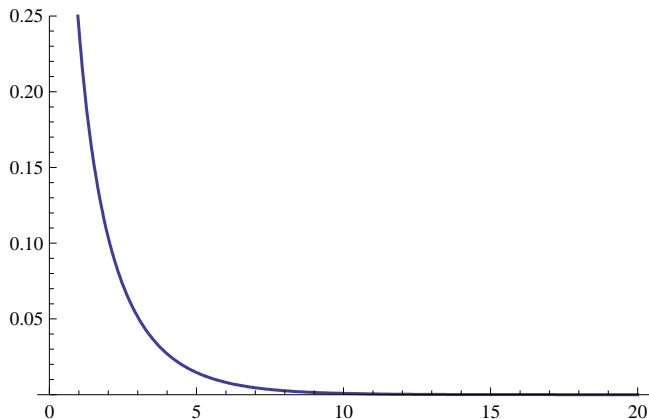
Definition (χ^2 degrees of freedom)

In a goodness-of-fit test, the number of **degrees of freedom** is one less than the number of cells.

- The χ^2 distribution has an associated degrees of freedom, just like the t distribution.
- Each χ^2 distribution has a slightly different shape, depending on the number of degrees of freedom.
- For example, we let χ^2_5 denote the chi-square statistic with 5 degrees of freedom.

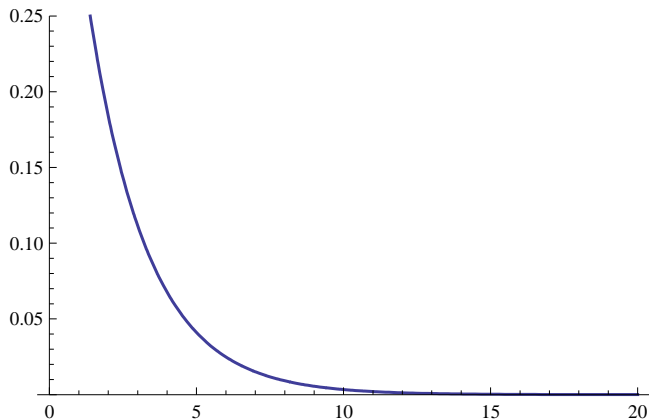
Chi-Square Degrees of Freedom

- The Graph of χ_1^2 .



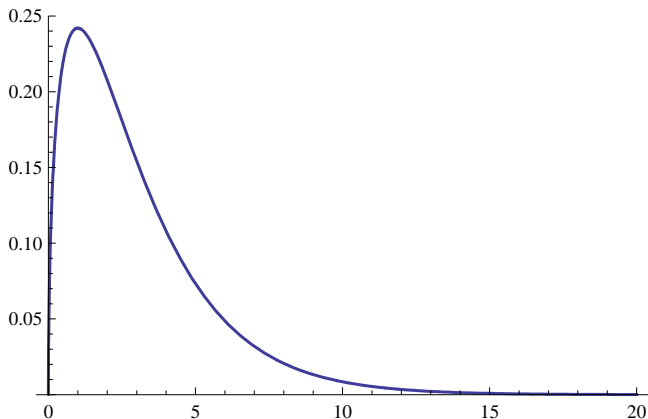
Chi-Square Degrees of Freedom

- The Graph of χ^2_2 .



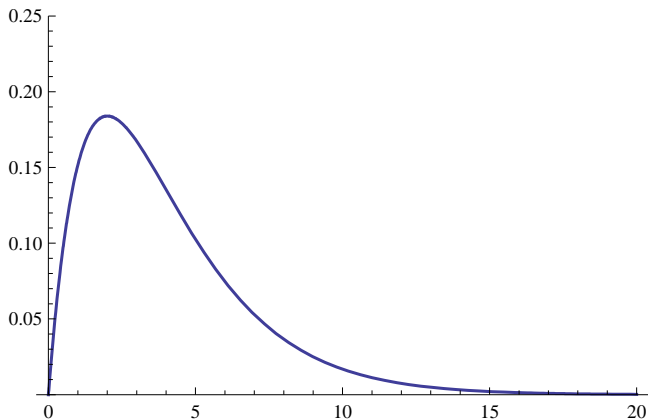
Chi-Square Degrees of Freedom

- The Graph of χ_3^2 .



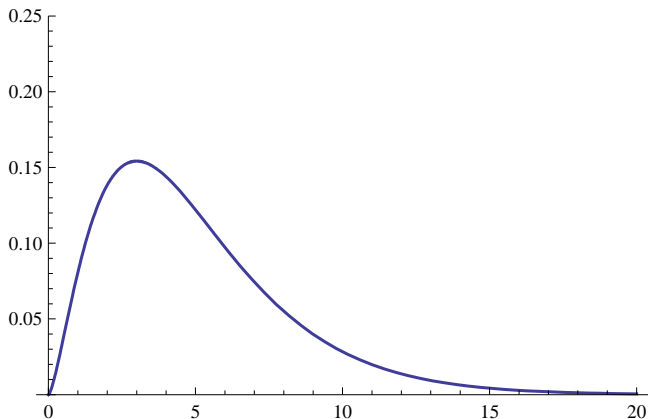
Chi-Square Degrees of Freedom

- The Graph of χ_4^2 .



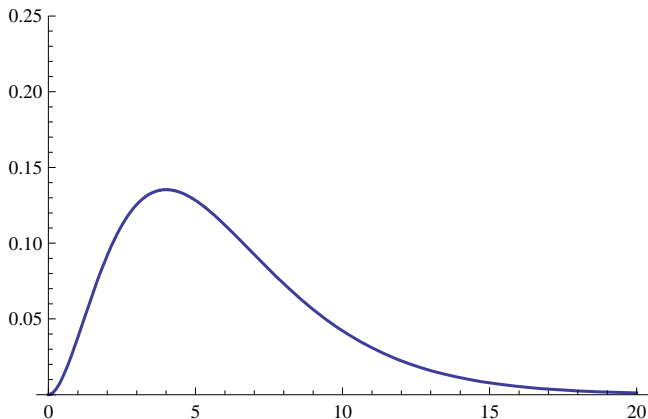
Chi-Square Degrees of Freedom

- The Graph of χ^2_5 .



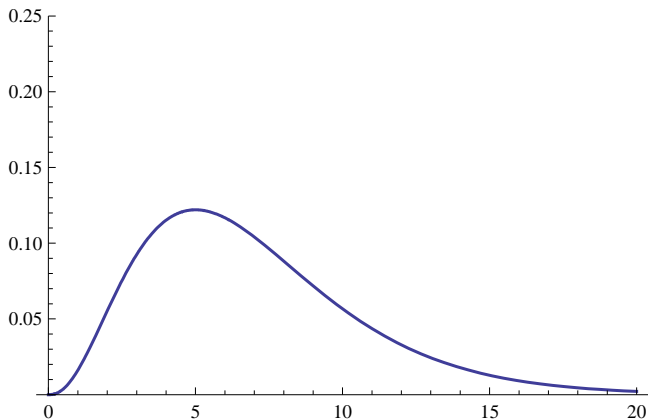
Chi-Square Degrees of Freedom

- The Graph of χ_6^2 .



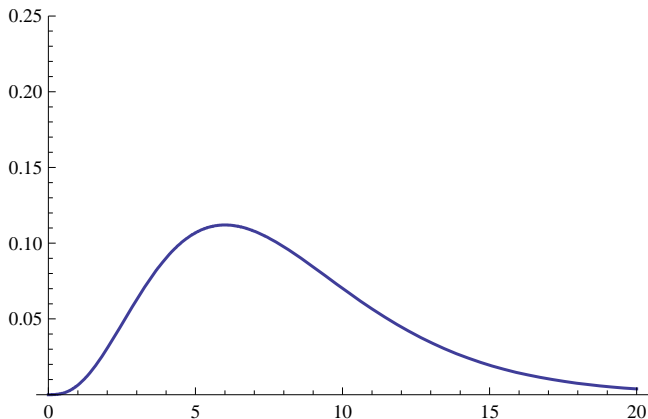
Chi-Square Degrees of Freedom

- The Graph of χ_7^2 .



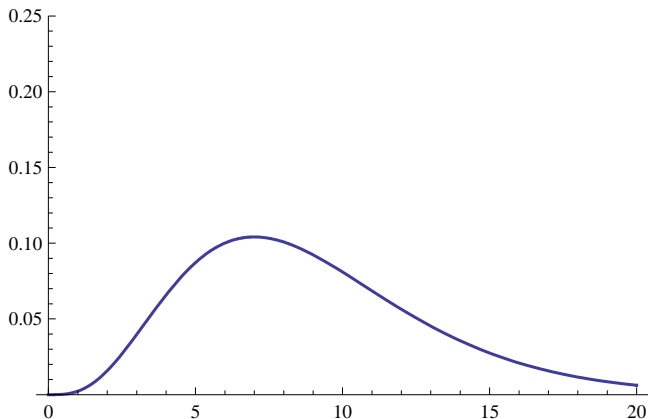
Chi-Square Degrees of Freedom

- The Graph of χ_8^2 .



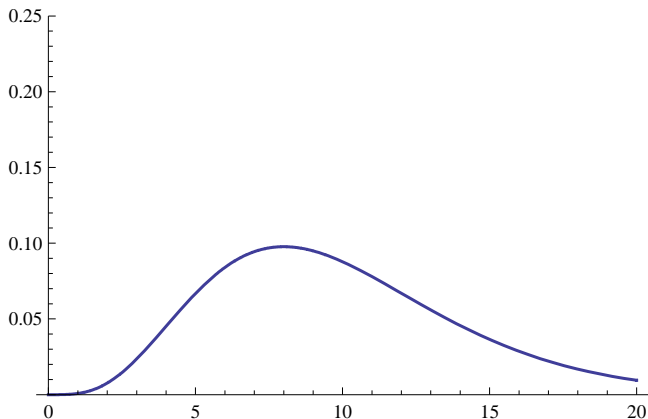
Chi-Square Degrees of Freedom

- The Graph of χ_9^2 .



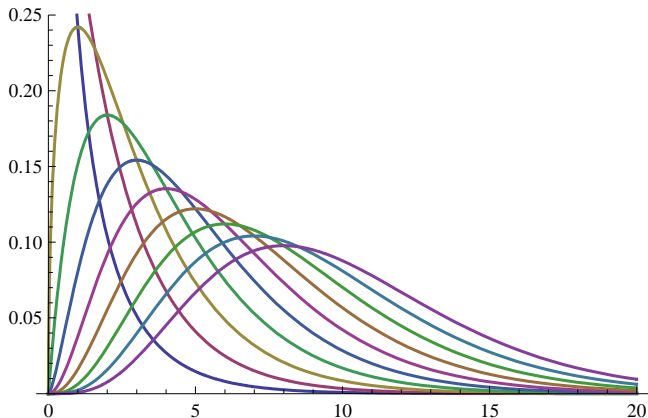
Chi-Square Degrees of Freedom

- The Graph of χ_{10}^2 .



Chi-Square Degrees of Freedom

- The Graphs of $\chi_1^2, \chi_2^2, \dots, \chi_{10}^2$.



Properties of χ^2

- The chi-square distribution with df degrees of freedom has the following properties.
 - $\chi^2 \geq 0$.
 - It is unimodal.
 - It is skewed right (not symmetric!)
 - $\mu_{\chi^2} = df$.
 - $\sigma_{\chi^2} = \sqrt{2df}$.
 - If df is large, then χ_{df}^2 is approximately normal with mean df and standard deviation $\sqrt{2df}$.

TI-83 Chi-square Probabilities

- Press `2nd DISTR`.
- Select χ^2 cdf.
- Enter the lower endpoint, the upper endpoint, and the degrees of freedom.
- Press `ENTER`. The probability appears in the display.

Practice

- Find $P(\chi_3^2 > 6)$.
- Find $P(4 < \chi_8^2 < 12)$.
- Find $P(\chi_5^2 < 10)$.
- Find $P(\chi_5^2 > 4.133)$.

The Goodness-of-Fit Test

- In our example, we found $\chi^2 = 4.133$.
- There are 6 categories (1 - 6), so there are 5 degrees of freedom.

The Goodness-of-Fit Test

Example (Steps 5, 6, and 7)

(5) $p\text{-value} = \chi^2_{\text{cdf}}(4.133, E99, 5) = 0.5304.$

(6) Accept $H_0.$

(7) The die is fair.

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Goodness-of-Fit Test on the TI-83

- Be careful when using the TI-83!
- There is a function called χ^2 -Test, but it does *not* perform the goodness-of-fit test.
- Some TI-84s have a GOF-Test function.
- The GOF-Test function does perform the goodness-of-fit test.

TI-83 Goodness-of-fit test

- Put the observed counts in list L_1 .
- Put the hypothetical proportions in list L_2 .
- Multiply L_2 by the sample size and store as L_3 . These are the expected counts.
- Calculate $(L_1 - L_3)^2 / L_3$ (either all at once or step by step).
- Go to `LIST > MATH` and select `sum` (item #5).
- Enter `Ans` and press `ENTER`. The value of χ^2 appears.
- Then use χ^2_{cdf} to find the p -value.

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One Last Comment

- It so happens that the die used in that example was not fair.
- The probability of each of 1, 2, 3, and 4 was 15% and the probability of each of 5 and 6 was 20%.
- Why did the test show that it was fair?

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Assignment

Homework

- Read Sections 14.1 - 14.2, pages 921 - 928.
- Let's Do It! 14.1.
- Exercises 1 - 5, page 928.